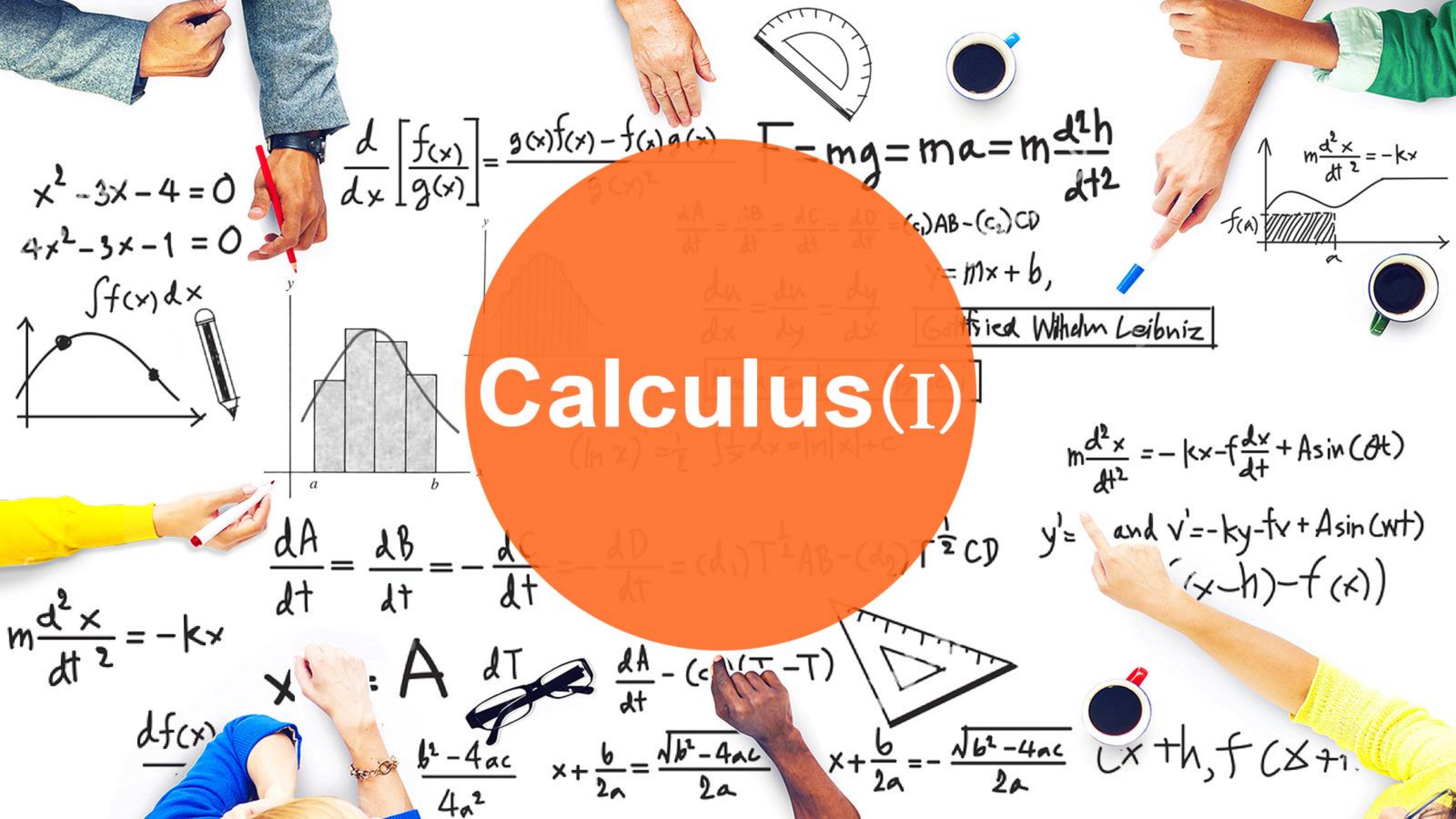
Calculus(Ⅰ)



Improper Integrals (II):

Infinite Integrands

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Problem Introduction

Proper integral: Integration function is bounded on [a,b].

extend

Find which point is unbounded?

Improper Integral: Integration function is unbounded on [a,b].

Problem Introduction

Considering the many complicated integrations that we have done , here is one that looks simple enough but is incorrect.

$$\int_{-2}^{1} \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_{-2}^{1} = -1 - \frac{1}{2} = -\frac{3}{2}$$

Our function, $f(x) = 1/x^2$, is not bounded at 0 point, so it is not integrable in the proper sense.

Definition 1

Let f be continuous on the half-open interval $[a,b)$

and suppose that $\lim_{x\to b^-}|f(x)|=\infty.$ Then

$$\int_a^b f(x)dx=\lim_{t\to b^-}\int_a^t f(x)dx$$

(1) If this limit exists and is finite,

in which case we say that the integral converges.

(2) Otherwise, this limit doesn't exist,

in which case we say that the integral diverges.

Definition 1 (Special point-Flaw point)

By N—L formula: $\int_a^b f(x)dx = [F(x)]\big|_a^b$ $(F'(x) = f(x))$, the improper integral

Consider flaw points: left end point, right end point, some middle point

Case 1

$f(x) \in C(a,b]$, a is a flaw point, we have

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a^+) = F(b) - \lim_{x \to a^+} F(x)$$

Case 2

$f(x) \in C[a,b)$, b is a flaw point, we have

$$\int_a^b f(x)\mathrm{d}x = F(b^-) - F(a) = \lim_{x \to b^-} F(x) - F(a)$$

Definition 2

Case 3

Let f be continuous on $[a, b]$ except at a number c,

where $a < c < b$, and suppose that $\lim_{x \to c} |f(x)| = \infty$.

Then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

(1) if both integrals on the right converge,

in which case, the improper integral converges.

(2) Otherwise, one or both don't converge

we say that $\int_a^b f(x)dx$ diverges.

Example 1

Evaluate, if possible, $\int_0^{a} \frac{dx}{\sqrt{a^2 - x^2}}$ $(a > 0)$.

$\because \lim_{x \to a^-} \frac{1}{\sqrt{a^2 - x^2}} = +\infty$ $\quad \therefore x = a$ is a flaw point,

$$\int_0^{a} \frac{dx}{\sqrt{a^2 - x^2}} = \lim_{t \to a^-} \int_0^{t} \frac{dx}{\sqrt{a^2 - x^2}} = \lim_{t \to a^-} \left[\arcsin \frac{x}{a} \right]_0^t$$

$$= \lim_{t \to a^-} \arcsin \frac{t}{a} - 0$$

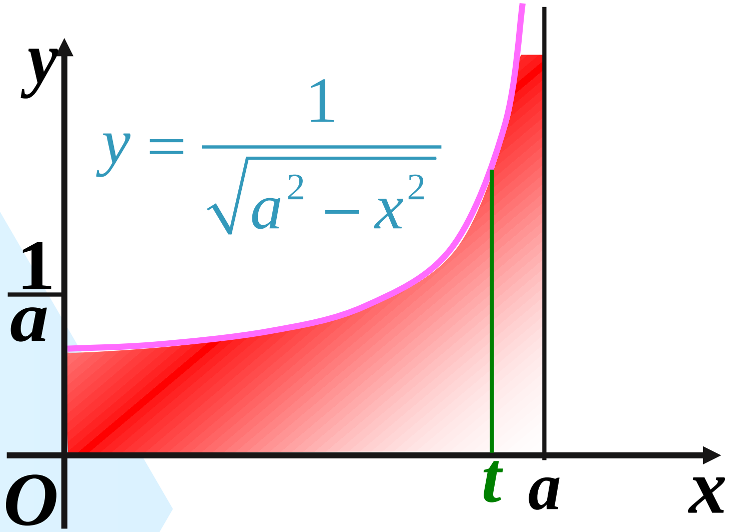
$$= \frac{\pi}{2}.$$

Q: Geometrical meaning?

Example 1

Evaluate:

$$\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} \quad (a > 0) = \frac{\pi}{2}.$$



Geometrical meaning of this improper integral is:

the area of locating below the curve $y = \frac{1}{\sqrt{a^2 - x^2}}$,

above the x-axis, between the lines $x = 0$ and $x = a$.

Example 2

Compute improper integral $\int_1^2 \frac{\mathrm{d}x}{x\ln x}$.

$$= \int_1^2 \frac{\mathrm{d}(\ln x)}{\ln x}$$

$$= \left[\ln(\ln x)\right]_1^2$$

$$= \ln(\ln 2) - \lim_{x \to 1^+} \ln(\ln x)$$

$$= \infty.$$

So the improper integral diverges.

Example 3

Prove the improper integral $\int_0^1 \frac{1}{x^q}\mathrm{d}x$, converges if $q<1$,

but diverges if $q \geq 1$.

(1) $q=1$, $\int_0^1 \frac{1}{x^q}\mathrm{d}x = \int_0^1 \frac{1}{x}\mathrm{d}x = [\ln x]_0^1 = \ln 1 - \lim_{x\to 0^+}(\ln x) = +\infty$

(2) $q \neq 1$, $\int_0^1 \frac{1}{x^q}\mathrm{d}x = \left[\frac{x^{1-q}}{1-q}\right]\Bigg|_{0^+}^{1} = \begin{cases} +\infty, & q>1, \\ \frac{1}{1-q}, & q<1. \end{cases}$

when $q<1$, converges, the value is $\frac{1}{1-q}$,

when $q \geq 1$, diverges.

Example 4

Find the value $\int_{-1}^{1} \frac{1}{x}\,dx.$

$\because \lim_{x\to 0} \frac{1}{x} = \infty \quad \therefore x = 0$ is flaw point. $\quad \because \int_{-1}^{1} \frac{1}{x}dx = \int_{-1}^{0} \frac{1}{x}dx + \int_{0}^{1} \frac{1}{x}dx$

$$\because \int_{0}^{1} \frac{1}{x}dx = \ln|x|\Big|_{0}^{1} = 0 - \lim_{x\to 0^{+}} \ln|x| = \infty \quad \text{diverges}$$

$$\therefore \int_{-1}^{1} \frac{1}{x}dx \quad \text{diverges}$$

$$\int_{-1}^{1} \frac{1}{x}dx = \ln|x|\Big|_{-1}^{1} = 0$$ ✗

Summary (Remember & Understand)

- $f(x) \in C(a,b]$, a is a flaw point, $F'(x) = f(x)$,

  $$\int_a^b f(x)\mathrm{d}x = F(b) - F(a^+) = F(b) - \lim_{x \to a^+} F(x)$$

- $f(x) \in C[a,b)$, b is a flaw point, $F'(x) = f(x)$,

  $$\int_a^b f(x)\mathrm{d}x = F(b^-) - F(a) = \lim_{x \to b^-} F(x) - F(a)$$

Questions and Answers

Evaluate $\int_0^1 \frac{1}{x} dx.$

$$= \lim_{t \to 0^+} \int_t^1 \frac{1}{x} dx$$

$$= \lim_{t \to 0^+} [\ln x]_t^1$$

$$= \ln 1 - \lim_{t \to 0^+} [\ln t]$$

$$= \infty.$$

Questions and Answers

Show that $\int_{-2}^{1} 1/x^2\,dx$ diverges.

$$\int_{-2}^{1} \frac{1}{x^2}dx = \int_{-2}^{0} \frac{1}{x^2}dx + \int_{0}^{1} \frac{1}{x^2}dx$$

The second integral on the right diverges by Eg3.

So the improper integral diverges.

Questions and Answers

Evaluate $\int_{0}^{+\infty} \frac{1}{x} dx.$

$$= \int_{0}^{1} \frac{1}{x} dx + \int_{1}^{+\infty} \frac{1}{x} dx$$

diverges

$$\therefore \int_{-\infty}^{+\infty} \frac{1}{x} dx$$ diverges

Infinite Integrands